

A POINTWISE BIPOLAR THEOREM

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ABSTRACT. We provide a pointwise bipolar theorem for \liminf -closed convex sets of positive Borel measurable functions on a σ -compact metric space without the assumption that the polar is a tight set of measures. As applications we derive a version of the transport duality under non-tight marginals, and a superhedging duality for semistatic hedging in discrete time.

1. INTRODUCTION

Given a dual pair of vector spaces $(X, Y, \langle \cdot, \cdot \rangle)$, the bipolar theorem states that every $\sigma(X, Y)$ -closed, convex set A with $0 \in A$ is equal to its bipolar $A^{\circ\circ}$. The result is a straightforward application of the Hahn-Banach separation theorem for locally convex topological vector spaces. Motivated by applications in mathematical finance, Brannath and Schachermayer [7] provide a version of the bipolar theorem on the cone $L_+^0(\Omega, \mathcal{F}, \mathbb{P})$ endowed with the topology induced by the convergence in probability which is not locally convex.

In the present work we focus on a pointwise version of the bipolar theorem of Brannath and Schachermayer when functions that are almost surely equal w.r.t. a reference measure are not identified. To that end, let $\mathcal{L}_+^0 = \mathcal{L}_+^0(\Omega)$ be the set of all Borel measurable functions $f: \Omega \rightarrow [0, +\infty]$, where Ω is a σ -compact metric space. Denoting by ca_+ the set of all finite positive Borel measures on Ω , we define $\langle f, \mu \rangle := \int f d\mu$ for all measurable f which are bounded from below and $\mu \in ca_+$. The polar and bipolar of a subset $H \subset \mathcal{L}_+^0$ are given by

$$H^\circ := \{\mu \in ca_+ : \langle f, \mu \rangle \leq 1 \text{ for all } f \in H\},$$

$$H^{\circ\circ} := \{f \in \mathcal{L}_+^0 : \langle f, \mu \rangle \leq 1 \text{ for all } \mu \in H^\circ\}.$$

A set $H \subset \mathcal{L}_+^0$ is called monotone, if $f \in H$ for all $f \in \mathcal{L}_+^0$ such that $f \leq h$ for some $h \in H$. Further, H is called \liminf -closed whenever $\liminf_n h_n \in H$ for every sequence (h_n) in H , and regular if $\sup_{h \in H \cap U_b} \langle h, \mu \rangle = \sup_{h \in H \cap C_b} \langle h, \mu \rangle$ for all $\mu \in ca_+$, or equivalently, if $(H \cap C_b)^\circ = (H \cap U_b)^\circ$. Here C_b and U_b denote the spaces of all bounded functions $f: \Omega \rightarrow \mathbb{R}$ that are continuous and upper semicontinuous, respectively. Then the following pointwise version of the bipolar theorem on \mathcal{L}_+^0 holds:

Theorem 1. *Let H be a nonempty monotone regular subset of \mathcal{L}_+^0 . Then $H = H^{\circ\circ}$ if and only if H is convex and closed under \liminf .*

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As an application we deduce the following regularity result:

Corollary 2. *Under the assumptions of Theorem 1 one has $H^\circ = (H \cap C_b)^\circ$.*

The proof of Theorem 1 is divided into two steps. We first present a bipolar theorem under an additional tightness assumption for \liminf -closed convex sets G of measurable functions that are bounded from below. In that case, it follows from the Choquet capacitability theorem (see e.g. [2, 8]) that the “superhedging” functional

$$\phi(f) := \inf\{m \in \mathbb{R} : m + g \geq f \text{ for some } g \in G\}$$

has a dual representation of the form $\phi(f) = \sup_{\mu \in ca_+} (\langle f, \mu \rangle - \phi^*(\mu))$, and the level sets $\{\phi^* \leq c\}$ are tight for all $c \in \mathbb{R}$, where ϕ^* denotes the convex conjugate. That G is equal to its bipolar $G^{\circ\circ}$ then follows from the representation of the superhedging functional. In a second step, we approximate $H \subset \mathcal{L}_+^0$ by a sequence of \liminf -closed convex sets (H_k) which satisfy the tightness assumption and thus by the first step have the bipolar representation $H_k = H_k^{\circ\circ}$. The \liminf -closedness is then used to show that $H = \bigcap_k H_k = (\bigcup_k H_k^\circ)^\circ = H^{\circ\circ}$.

In particular, Theorem 1 implies that $H \cap \mathcal{L}^\infty$ is $\sigma(\mathcal{L}^\infty, ca)$ -closed, where \mathcal{L}^∞ is the set of all bounded Borel measurable functions $f : \Omega \rightarrow \mathbb{R}$. In case that functions in \mathcal{L}^∞ are identified if they are equal almost surely w.r.t. a reference measure, it follows from the Krein-Smulian theorem that a convex set in \mathcal{L}^∞ is weak*-closed if it is Fatou closed, i.e. closed under bounded almost surely convergent sequences. If the dominating measure is replaced by a capacity, a similar result is shown in [16], however under the assumption that the capacity allows for an essential infimum.

Finally, we give two applications of the pointwise bipolar theorem. The first one is a transport duality with non-tight marginals. In the classical transport problem one optimizes $\langle f, \mu \rangle$ for a given function $f \in \mathcal{L}_+^0$ over the set of all measures μ with prescribed marginals. Motivated by the hedging problem in mathematical finance, we consider the modified version where $\langle f, \mu \rangle$ is optimized over all measures where the marginals are in given non-tight sets H_i° . By means of Theorem 1 we identify the modified transport problem with a corresponding superhedging functional. As a second application, we consider the problem of pointwise superreplicating a path-dependent contingent claim f by investing dynamically and statically at the terminal time, i.e. minimizing the hedging costs $\varphi(g)$ over the trading strategies (ϑ, g) such that $f(S_1, \dots, S_T) \leq (\vartheta \cdot S)_T + g(S_T)$. Here $(\vartheta \cdot S)_T$ denotes the discrete time stochastic integral and φ is a (sublinear) pricing functional for the plain vanilla option $g(S_T)$. The bipolar theorem is then used to show the superhedging duality. This is classical problem in mathematical finance and was investigated e.g. in [1, 10, 11, 12] though in different setting, i.e. either in continuous time or under the assumption that a reference measure exists.

The paper is organized as follows. In Section 2 and Section 3 we state our main results. Their applications to the transport problem and the robust hedging problem are given in Section 4 and Section 5, respectively.

2. A BIPOLAR THEOREM FOR \liminf -CLOSED SETS

Let Ω be a metric space. Denote by \mathcal{L}_{b-}^0 the space of all Borel measurable functions $f : \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$ which are bounded from below. Let ca_+ be the set of all finite positive Borel measures on Ω , including the subset ca_+^1 of all probability

measures. Define $\langle f, \mu \rangle := \int f d\mu$ for all $f \in \mathcal{L}_{b-}^0$ and $\mu \in ca_+$. The polar and bipolar sets of $H \subset \mathcal{L}_{b-}^0$ are given by

$$H^\circ := \{\mu \in ca_+ : \langle f, \mu \rangle \leq 1 \text{ for all } f \in H\}$$

and

$$H^{\circ\circ} := \{f \in \mathcal{L}_{b-}^0 : \langle f, \mu \rangle \leq 1 \text{ for all } \mu \in H^\circ\}.$$

Let C_b and U_b be the sets of all bounded functions $f : \Omega \rightarrow \mathbb{R}$ that are continuous and upper semicontinuous, respectively.

Definition 3. We say that a subset H of \mathcal{L}_{b-}^0 is

- *monotone*, if $f \in H$ for all $f \in \mathcal{L}_{b-}^0$ such that $f \leq h$ for some $h \in H$,
- *nontrivial*, if $H \neq \emptyset$ and $H \neq \mathcal{L}_{b-}^0$,
- *normalized*, if $0 \in H$ and $\varepsilon \notin H$ for every $\varepsilon > 0$,
- *tight*, if for every $m \in \mathbb{R}$ with $m \in H$, $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a compact set $K \subset \Omega$ such that $m - \varepsilon + n1_{K^c} \in H$,
- *closed under \liminf* , if $\liminf_n h_n \in H$ for every sequence (h_n) in H with $h_n \geq c$ for some $c \in \mathbb{R}$,
- *regular*, if $\sup_{h \in H \cap U_b} \langle h, \mu \rangle = \sup_{h \in H \cap C_b} \langle h, \mu \rangle$ for all $\mu \in ca_+^1$.

For a normalized and monotone set $H \subset \mathcal{L}_{b-}^0$ it suffices to restrict to $m = 0$ in the definition of tightness. Further, if Ω is compact, every monotone set $H \subset \mathcal{L}_{b-}^0$ is automatically tight.

Lemma 4. Suppose that H is monotone and closed under \liminf . Then H is nontrivial if and only if $H - m$ is normalized for a unique $m \in \mathbb{R}$.

Proof. Obviously, if $H - m$ is normalized for some m , then H is nontrivial. Conversely, if H is nontrivial, define

$$M = \{m \in \mathbb{R} : h \geq m \text{ for some } h \in H\}.$$

Then $M \neq \emptyset$ and monotonicity of H implies that M is bounded from above, i.e. $m := \sup M \in \mathbb{R}$. Let (m_n) in M such that $m_n \uparrow m$. By definition, there exists (h_n) in H such that $h_n \geq m_n$. Since $h_n \geq m_1$ and H is closed under \liminf , one has $m \leq \liminf_n h_n \in H$. Hence $m \in H$, which shows that $H - m$ is normalized. \square

Remark 5. Every monotone subset H of \mathcal{L}_{b-}^0 is closed under \liminf if and only if $\sup_n h_n \in H$ for every increasing sequence (h_n) in H . Indeed, if (h_n) is a sequence in H which is bounded from below by a constant, then $\liminf h_n = \sup_n g_n$ for $g_n := \inf_{m \geq n} h_m$ which is an element of H by monotonicity.

Proposition 6. Let H be a monotone normalized regular tight subset of \mathcal{L}_{b-}^0 . Then, $H = H^{\circ\circ}$ if and only if H is convex and closed under \liminf .

Proof. If $H = H^{\circ\circ}$ then H is convex and closed under \liminf by Fatou's lemma. Conversely, suppose H is convex and closed under \liminf . Define

$$\phi(f) := \inf\{m \in \mathbb{R} : m + h \geq f \text{ for some } h \in H\}$$

for all $f \in \mathcal{L}_{b-}^0$. Then, one has $\phi(f + m) = \phi(f) + m$ for every $f \in \mathcal{L}_{b-}^0$ and $m \in \mathbb{R}$, and $\phi(0) = 0$ by normalization of H . Moreover, ϕ is increasing and convex, by monotonicity and convexity of H . Our goal is to apply the Choquet capacitability theorem. To that end, let (f_n) be a sequence in C_b which decreases pointwise to 0. By tightness of H , for every $\varepsilon > 0$ there exists a compact $K \subset \Omega$ such that

$\|f_1\|_\infty 1_{K^c} - \varepsilon \in H$, where $\|\cdot\|_\infty$ denotes the supremum norm. It follows from Dini's lemma that $f_n 1_K \leq \varepsilon$ for n large enough, so that $f_n \leq 2\varepsilon + \|f_1\|_\infty 1_{K^c} - \varepsilon$, and therefore $\phi(f_n) \leq 2\varepsilon$. As $\varepsilon > 0$ was arbitrary it follows that $\lim_n \phi(f_n) = 0$. Define

$$\phi_C^*(\mu) := \sup_{f \in C_b} (\langle f, \mu \rangle - \phi(f)) \quad \text{and} \quad \phi_U^*(\mu) := \sup_{f \in U_b} (\langle f, \mu \rangle - \phi(f))$$

for all $\mu \in ca_+$. Observe that $\phi_C^*(\mu) = \phi_U^*(\mu) = +\infty$ whenever μ is not a probability measure, because $\phi(m) = m$ for each $m \in \mathbb{R}$. Further, one has

$$\phi_C^*(\mu) = \sup_{f \in H \cap C_b} \langle f, \mu \rangle$$

for every $\mu \in ca_+^1$. Indeed, by definition the left hand side is larger than the right hand side. To show the other inequality, fix $f \in C_b$ and $\varepsilon > 0$. By definition of ϕ there exists $h \in H$ such that $\phi(f) + \varepsilon + h \geq f$. For $f' := f - \phi(f) - \varepsilon$, one has $f' \in H \cap C_b$ because $f' \leq h$, and

$$\langle f', \mu \rangle = \langle f, \mu \rangle - \phi(f) - \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, the statements holds. Similarly, it follows that $\phi_U^*(\mu) = \sup_{f \in H \cap U_b} \langle f, \mu \rangle$, so that by regularity

$$\phi_C^*(\mu) = \sup_{f \in H \cap C_b} \langle f, \mu \rangle = \sup_{f \in H \cap U_b} \langle f, \mu \rangle = \phi_U^*(\mu).$$

We next show that ϕ is continuous from below on \mathcal{L}_{b-}^0 . Let (f_n) be a sequence in \mathcal{L}_{b-}^0 which increases pointwise to $f \in \mathcal{L}_{b-}^0$. Since ϕ is increasing, one has $\phi(f) \geq \lim_n \phi(f_n)$. As for the other inequality, we assume that $\lim_n \phi(f_n) < \infty$ since otherwise the statement is obvious. For each n , fix $m_n \in \mathbb{R}$ and $h_n \in H$ such that

$$m_n \leq \phi(f_n) + 1/n \quad \text{and} \quad m_n + h_n \geq f_n.$$

Note that the sequence (m_n) has a limit. Since $h_n \geq f_n - m_n \geq c$ for some $c \in \mathbb{R}$ and H is closed under \liminf , it follows that $h := \liminf_n h_n \in H$. Hence

$$\lim_n m_n + h = \liminf_n (m_n + h_n) \geq \liminf_n f_n = f$$

which shows that $\phi(f) \leq \lim_n \phi(f_n)$. Moreover, we obtain $\phi(f) \leq 0$ if and only if $f \leq h$ for some $h \in H$ by applying this argumentation to the constant sequence $f_n := f$ for all $n \in \mathbb{N}$ which, by monotonicity, shows that $H = \{f \in \mathcal{L}_{b-}^0 : \phi(f) \leq 0\}$.

An application of the representation results in [2] modified to the cone \mathcal{L}_{b-}^0 yields

$$(1) \quad \phi(f) = \sup_{\mu \in ca_+^1} (\langle f, \mu \rangle - \phi_C^*(\mu))$$

for all $f \in \mathcal{L}_{b-}^0$, with the convention $+\infty - \infty := -\infty$.

Finally we show that $H = H^{\circ\circ}$. Obviously, $H \subset H^{\circ\circ}$. As for the other inclusion, fix $f \notin H$. Since $\phi(f) > 0$ it follows from (1) that there exists $\mu \in ca_+^1$ such that

$$(2) \quad \langle f, \mu \rangle > \phi_C^*(\mu) \geq 0.$$

Further, since $\phi(h) \leq 0$ for every $h \in H$, it follows again from (1) that $\phi_C^*(\mu) \geq \langle h, \mu \rangle$ for every $h \in H$. Hence, by scaling (2) there exists $\mu' \in ca_+$ such that

$$\langle f, \mu' \rangle > 1 \geq \langle h, \mu' \rangle \quad \text{for all } h \in H.$$

This shows that $\mu' \in H^\circ$, and therefore $f \notin H^{\circ\circ}$. \square

Corollary 7. *For every monotone convex regular tight set $H \subset \mathcal{L}_{b-}^0$ which is closed under \liminf and $0 \in H$, one has $H = H^{\circ\circ}$.*

Proof. If $H = \mathcal{L}_{b-}^0$ the statement obviously holds. Otherwise, H is nontrivial and by Lemma 4 there exists $m \in \mathbb{R}$ such that $\tilde{H} := H - m$ is centered. Following the arguments in the proof of Proposition 6, it follows that $f \in \tilde{H}$ if and only if $\langle f, \mu \rangle \leq \alpha_{\tilde{H}}(\mu)$ for all $\mu \in ca_+^1$, where $\alpha_{\tilde{H}}(\mu) = \sup_{f \in \tilde{H} \cap C_b} \langle f, \mu \rangle$ is the support function of $\tilde{H} \cap C_b$. Hence, $f \in H$ if and only if $f - m \in \tilde{H}$ if and only if $\langle f, \mu \rangle \leq \alpha_{\tilde{H}}(\mu) + m = \alpha_H(\mu)$ for all $\mu \in ca_+^1$. Since $0 \in H$ we can apply the scaling argument in the end of the proof of Proposition 6, which implies $H = H^{\circ\circ}$. \square

3. THE PROOF OF THEOREM 1

Throughout this section we assume that Ω is a σ -compact metric space, that is, there exists a sequence (K_n) of compact subsets of Ω such that $\Omega = \bigcup_n K_n$. For $H \subset \mathcal{L}_+^0$ the bipolar $H^{\circ\circ} = \{f \in \mathcal{L}_+^0 : \langle f, \mu \rangle \leq 1 \text{ for all } \mu \in H^\circ\}$ is a subset of \mathcal{L}_+^0 , while for $G \subset \mathcal{L}_{b-}^0$ the bipolar $G^{\circ\circ} = \{f \in \mathcal{L}_{b-}^0 : \langle f, \mu \rangle \leq 1 \text{ for all } \mu \in G^\circ\}$ is a subset of \mathcal{L}_{b-}^0 . Recall that $H^\circ = \{\mu \in ca_+ : \langle f, \mu \rangle \leq 1 \text{ for all } f \in H\}$. In the following we provide the proof of the main result.

Proof of Theorem 1. If $H = H^{\circ\circ}$ then H is convex and closed under \liminf by Fatou's lemma. As for the other implication we can assume that $H \neq \mathcal{L}_+^0$ because otherwise $H = H^{\circ\circ}$ obviously holds. Let (K_n) be an increasing sequence of compact subsets of Ω such that $\Omega = \bigcup_n K_n$, and define the function $\gamma: \Omega \rightarrow [0, +\infty)$ by $\gamma := \sum_n 1_{K_n^c}$. Then, for every $c \in \mathbb{R}_+$, the level set $\{\gamma \leq c\}$ is compact. We claim that H_k is nontrivial for k large enough, where H_k is given by

$$H_k := \{f \in \mathcal{L}_{b-}^0 : f \leq h + \gamma/k \text{ for some } h \in H\}$$

for all $k \in \mathbb{N}$. Indeed, if H_k is trivial for every k , then there exists $h_k \in H$ such that $k \leq h_k + \gamma/k$ for all k . However, since H is closed under \liminf , this implies that $h = \liminf_k h_k = +\infty \in H$, in contradiction to $H \neq \mathcal{L}_+^0$. Further, H_k is closed under \liminf for each k , since for every sequence (f_n) in H_k with $c \leq f_n \leq h_n + \gamma/k$ for $h_n \in H$ and $c \in \mathbb{R}$, one has $\liminf_n f_n \leq h + \gamma/k$ for $h = \liminf_n h_n \in H$. By Lemma 4 it follows that $H_k - m_k$ is normalized for a unique $m_k \in \mathbb{R}$. In particular, there exists $h_k \in H$ such that $m_k \leq h_k + \gamma/k$. For $n \in \mathbb{N}$ define the compact set $K := \{\gamma \leq k(m_k + n)\}$. Then

$$m_k + n1_{K^c} \leq h_k + \gamma/k$$

so that $n1_{K^c} \in H_k - m_k$, that is, $H_k - m_k$ is tight. Since γ is lower semicontinuous, there exists a sequence (γ_n) of continuous bounded functions such that $0 \leq \gamma_n \uparrow \gamma/k$. For every $\mu \in ca_+^1$ one has

$$\begin{aligned} (3) \quad \sup_{f \in H_k \cap C_b} \langle f, \mu \rangle &\geq \sup_{f \in H \cap C_b} \sup_{n \in \mathbb{N}} \langle f + \gamma_n, \mu \rangle = \sup_{f \in H \cap C_b} \langle f, \mu \rangle + \langle \gamma/k, \mu \rangle \\ &= \sup_{f \in H \cap U_b} \langle f, \mu \rangle + \langle \gamma/k, \mu \rangle \geq \sup_{f \in H_k \cap U_b} \langle f, \mu \rangle \end{aligned}$$

where we have to justify the last inequality. To that end, fix $f \in H_k \cap U_b$ so that $f \leq h + \gamma/k$ for some $h \in H$. Then $0 \vee (f - \gamma/k) \leq h$, so that $0 \vee (f - \gamma/k) \in H \cap U_b$ by monotonicity of H , which implies that $\langle f, \mu \rangle \leq \langle 0 \vee (f - \gamma/k), \mu \rangle + \langle \gamma/k, \mu \rangle$. Since $H_k \cap C_b \subset H_k \cap U_b$, it follows from (3) that H_k is regular.

In summary, $H_k - m_k$ is a monotone convex normalized regular tight subset of \mathcal{L}_{b-}^0 . By Corollary 7 one has $H_k = H_k^{\circ\circ}$. Since γ is positive, it follows that $H \subset H_k \cap \mathcal{L}_+^0$ for every k . On the other hand, if $f \in H_k \cap \mathcal{L}_+^0$ for every k , then there exists a sequence (h_k) in H such that $f \leq h_k + \gamma/k$. But then $f \leq h := \liminf_k h_k$, and since H is monotone and closed under \liminf , it follows that $f \in H$. Thus, one has

$$H = \bigcap_k (H_k \cap \mathcal{L}_+^0) = \bigcap_k (H_k^{\circ\circ} \cap \mathcal{L}_+^0) = \left(\bigcup_k H_k^\circ \right)^\circ \cap \mathcal{L}_+^0.$$

Since $\bigcup_k H_k^\circ \subset H^\circ$, it follows that $H = (\bigcup_k H_k^\circ)^\circ \cap \mathcal{L}_+^0 \supset H^{\circ\circ}$. On the other hand $H \subset H^{\circ\circ}$ always holds, so that $H = H^{\circ\circ}$ and the proof is complete. \square

Proof of Corollary 2. By definition it holds $H^\circ \subset (H \cap C_b)^\circ$. As for the other inclusion, fix $\mu \in ca_+$ such that $\mu \notin H^\circ$. Then, by definition, there exists $h \in H$ such that $\langle h, \mu \rangle > 1$. Further we may assume that h is bounded since $h \wedge n \leq h$, and monotonicity of H implies that $h \wedge n \in H$ for every n . Moreover the measure μ is tight since $\mu(K_n^c) \downarrow \mu(\emptyset) = 0$, and therefore inner regular. In particular there exists a sequence (h_n) of upper semicontinuous function such that $0 \leq h_n \leq h$ and $\langle h_n, \mu \rangle \rightarrow \langle h, \mu \rangle$. Hence $\langle h_{n_0}, \mu \rangle > 1$ which implies that $\mu \notin (H \cap U_b)^\circ = (H \cap C_b)^\circ$, where the last equality holds by assumption. Thus indeed $H^\circ = (H \cap C_b)^\circ$. \square

4. A TRANSPORT DUALITY WITH NON-TIGHT MARGINALS

Let Ω_1 and Ω_2 be two σ -compact metric spaces and fix two nonempty monotone and convex sets $H_i \subset \mathcal{L}_+^0(\Omega_i)$. For $i = 1, 2$, we assume that H_i is regular and closed under \liminf . It follows from Theorem 1 that

$$H_i = \{f \in \mathcal{L}_+^0(\Omega_i) : \langle f, \mu \rangle \leq 1 \text{ for } \mu \in H_i^\circ\} = \{f \in \mathcal{L}_+^0(\Omega_i) : \pi_i(f) \leq 1\}$$

where the functional $\pi_i : \mathcal{L}_+^0(\Omega_i) \rightarrow [0, +\infty]$ is given by

$$\pi_i(f) := \sup_{\mu \in H_i^\circ} \langle f, \mu \rangle.$$

On the space $\Omega := \Omega_1 \times \Omega_2$ we consider the product topology. For $h_i \in H_i$, $i = 1, 2$, we write $h_1 \oplus h_2 : \Omega \rightarrow [0, +\infty]$ for the function $h_1 \oplus h_2(\omega) := h_1(\omega_1) + h_2(\omega_2)$. Define the set

$$H := \{f \in \mathcal{L}_+^0(\Omega) : f \leq h_1 \oplus h_2 \text{ for } h_i \in \mathcal{L}_+^0(\Omega_i) \text{ with } \pi_1(h_1) + \pi_2(h_2) \leq 1\}.$$

For a measure $\mu \in ca_+(\Omega)$ denote by $\mu_1 := \mu(\cdot \times \Omega_2) \in ca_+(\Omega_1)$ and $\mu_2 := \mu(\Omega_1 \times \cdot) \in ca_+(\Omega_2)$ its marginal distributions.

Theorem 8. *Suppose there exist measures $\mu_i^* \in ca_+(\Omega_i)$ such that $\mu_i \ll \mu_i^*$ for all $\mu_i \in H_i^\circ$, for $i = 1, 2$. Then one has*

$$H = \{f \in \mathcal{L}_+^0(\Omega) : \pi(f) \leq 1\}$$

where the functional $\pi : \mathcal{L}_+^0(\Omega) \rightarrow [0, +\infty]$ is given by

$$(4) \quad \pi(f) := \sup\{\langle f, \mu \rangle : \mu \in ca_+(\Omega) \text{ such that } \mu_1 \in H_1^\circ, \mu_2 \in H_2^\circ\}.$$

In particular, it holds $H = H^{\circ\circ}$, where $H^\circ = \{\mu \in ca_+(\Omega) : \mu_1 \in H_1^\circ, \mu_2 \in H_2^\circ\}$.

Remarks 9.

1. If each H_i° consists of exactly one probability measure ν_i , the optimization problem (4) reduces to the Kantorovich transport problem, see [14]. For corresponding duality results we refer to [6, 15, 20] and the references therein, see also [4, 5] for the related martingale transport duality.
2. It follows from Theorem 8 that for each $f \in \mathcal{L}_+^0(\Omega)$ the following superhedging duality holds:

$$\phi(f) := \inf \left\{ m \in \mathbb{R}_+ : f \leq f_1 \oplus f_2 \text{ for } f_i \in \mathcal{L}_+^0(\Omega_i) \text{ with } \sum_i \pi_i(f_i) \leq m \right\} = \pi(f).$$

Indeed, for every $m > 0$, one has $f/m \in H$ if and only if $f \leq f_1 \oplus f_2$ for $f_i \in \mathcal{L}_+^0(\Omega_i)$ with $\sum_i \pi_i(f_i) \leq m$, so that $\phi(f) \leq m$ whenever $\pi(f) \leq m$. In financial terms, the seller of a contingent claim f protects himself against losses by optimally investing in the traded derivatives f_1 and f_2 with ask prices $\pi_i(f_i)$. While for the Kantorovich transport problem the pricing functionals $\pi_i(f_i) = \langle f_i, \nu_i \rangle$ are linear, in [3] the pricing rules π_i , $i = 1, 2$, are assumed to be sublinear reflecting market incompleteness. However, the pricing rules in [3] are continuous from above, i.e. the marginals H_i° are assumed to be tight.

3. If both H_i are such that $\lim \text{med}_n h_i^n \in H$ for every sequence (h_i^n) in H_i , then the assumption that H_i° are dominated is not needed (for the concept of medial limits we refer to the next section). In that case, one replaces $h_i := \limsup_n h_i^n$ by $h_i := \lim \text{med}_n h_i^n$ in the following proof.

Proof. The goal is to apply Theorem 1. It is clear that H is nonempty, monotone, and convex. Moreover, for every $\mu \in ca_+(\Omega)$ one has

$$(5) \quad \sup_{h \in H \cap C_b(\Omega)} \langle h, \mu \rangle = \max_{i=1,2} \sup_{h_i \in H_i} \langle h_i, \mu_i \rangle,$$

and the same holds true if $H \cap C_b(\Omega)$ is replaced by H . Indeed, since every $h \in H$ satisfies $h \leq h_1 \oplus h_2$ for $h_i \in \mathcal{L}_+^0(\Omega_i)$, $i = 1, 2$, with $\pi_1(h_1) + \pi_2(h_2) \leq 1$, it follows that

$$\langle h, \mu \rangle \leq \langle h_1 \oplus h_2, \mu \rangle = \sum_i \langle h_i, \mu_i \rangle \leq \sum_i \pi_i(h_i) \sup_{f \in H_i} \langle f, \mu_i \rangle \leq \max_i \sup_{f \in H_i} \langle f, \mu_i \rangle,$$

because $h_i/\pi_i(h_i) \in H_i$ so that $\langle h_i, \mu_i \rangle \leq \pi_i(h_i) \sup_{f \in H_i} \langle f, \mu_i \rangle$ (with the convention $0 \cdot (+\infty) = +\infty$). This shows that the right hand side of (5) is greater than the left hand side. As for the other inequality, assume without loss of generality that maximum on the right hand side is attained at $i = 1$. By Corollary 2 one has $\sup_{f \in H_1} \langle f, \mu_1 \rangle = \sup_{f \in H_1 \cap C_b(\Omega_1)} \langle f, \mu_1 \rangle$, so that for every $\varepsilon > 0$ there exists $h_1 \in H_1 \cap C_b(\Omega_1)$ which satisfies $\max_i \sup_{f \in H_i} \langle f, \mu_i \rangle \leq \langle h_1, \mu_1 \rangle + \varepsilon$. Define the function $h \in H \cap C_b(\Omega)$ by $h(\omega) := h_1(\omega_1)$. Then, since $\langle h, \mu \rangle = \langle h_1, \mu_1 \rangle$, it follows that $\max_i \sup_{h_i \in H_i} \langle h_i, \mu_i \rangle \leq \sup_{h \in H \cap C_b(\Omega)} \langle h, \mu \rangle$. In particular, one has

$$H^\circ = \{ \mu \in ca_+(\Omega) \text{ such that } \mu_1 \in H_1^\circ, \mu_2 \in H_2^\circ \}.$$

We are left to show that H is closed under \liminf . Fix an increasing sequence (h^n) in H . Then $h^n \leq h_1^n \oplus h_2^n$ for $h_i^n \in \mathcal{L}_+^0(\Omega_i)$ with $\pi_1(h_1^n) + \pi_2(h_2^n) \leq 1$. Since $h_i^n \geq 0$, we can apply the Komlós' theorem (see [9, Lemma A.1]) to obtain forward convex combinations $\tilde{h}_i^n \in \text{conv}\{h_i^k : k \geq n\}$ which have a μ_i^* -almost sure limit. Define $h_i := \limsup_n \tilde{h}_i^n \in \mathcal{L}_+^0$, so that $\mu_i^*(h_i = \liminf_n \tilde{h}_i^n) = 1$. By the bipolar representation of H_i and Fatou's lemma, it follows that $h_i \in H_i$. Moreover, we

obtain

$$\pi_1(h_1) + \pi_2(h_2) \leq \liminf_n (\pi_1(\tilde{h}_1^n) + \pi_2(\tilde{h}_2^n)) \leq \liminf_n (\pi_1(h_1^n) + \pi_2(h_2^n)) \leq 1$$

again by Fatou's lemma and convexity of π_i . But then

$$\sup_n h^n = \liminf_n \tilde{h}^n \leq \liminf_n (\tilde{h}_1^n \oplus \tilde{h}_2^n) \leq \limsup_n (\tilde{h}_1^n \oplus \tilde{h}_2^n) \leq h_1 \oplus h_2,$$

which shows that $h \in H$. Therefore,

$$H = \{h \in \mathcal{L}_+^0(\Omega) : \langle h, \mu \rangle \leq 1 \text{ for } \mu \in H^\circ\} = \{h \in \mathcal{L}_+^0(\Omega) : \pi(f) \leq 1\},$$

where the first equality follows from Theorem 1. \square

5. ROBUST HEDGING IN DISCRETE TIME

Given a time horizon $T \in \mathbb{N}$, we consider the state space $\Omega := \mathbb{R}_{++}^T := (0, +\infty)^T$, and denote by $S_t: \Omega \rightarrow \mathbb{R}_{++}$ the projection on the t -th coordinate $S_t(\omega) = \omega_t$. We assume that the canonical process $(S_t)_{t=1, \dots, T}$ describes the discounted price process of a financial asset. We consider an agent who is allowed to invest dynamically in this asset and statically in a plain vanilla option on S_T . Thus, the set of trading strategies Θ consists of pairs (ϑ, g) where $\vartheta = (\vartheta_2, \dots, \vartheta_T)$ and each $\vartheta_t: \mathbb{R}_{++}^{t-1} \rightarrow \mathbb{R}$ is universally measurable, $g: \mathbb{R}_{++} \rightarrow \mathbb{R} \cup \{+\infty\}$ is a Borel measurable function which is bounded from below and satisfies $\varphi(g) \leq 0$. Here $\varphi(g)$ denotes the price of the plain vanilla option $g(S_T)$, given by the pricing functional

$$\varphi(g) := \sup_{\mu \in \mathcal{Q}} \langle g, \mu \rangle,$$

where \mathcal{Q} is a set of probability measures on \mathbb{R}_{++} . We assume that \mathcal{Q} is nonempty, convex, and compact w.r.t. the weak topology induced by the continuous bounded functions on \mathbb{R}_{++} . The outcome of the trading strategy $(\vartheta, g) \in \Theta$ is the universally measurable function $(\vartheta \cdot S)_T + g(S_T): \Omega \rightarrow \mathbb{R} \cup \{+\infty\}$, where $g(S_T)(\omega) := g(S_T(\omega))$ and

$$(\vartheta \cdot S)_T(\omega) := \sum_{t=2}^T \vartheta_t(S_1(\omega), \dots, S_{t-1}(\omega)) (S_t(\omega) - S_{t-1}(\omega)).$$

In the following we make use of so-called medial limits, see [17, 18]. A medial limit is a positive linear functional $\lim_{\text{med}}: l^\infty \rightarrow \mathbb{R}$ which satisfies $\liminf \leq \lim_{\text{med}} \leq \limsup$ and $\omega \mapsto f(\omega) := \lim_{\text{med}} f_n(\omega)$ is universally measurable for every bounded sequence of universally measurable functions (f_n) . We assume that a medial limit exists, which for instance is guaranteed under the usual axioms of ZFC and Martin's Axiom. For a discussion of the medial limit as a tool for pointwise convex optimization problems we refer to [2], and as a tool for the aggregation of stochastic integrals to [19].

Proposition 10. *Assume that $\lim_{k \rightarrow \infty} \varphi((id - k)^+) = 0$, there exists $\mu^* \in \mathcal{Q}$ such that $\mu \ll \mu^*$ for all $\mu \in \mathcal{Q}$, and the smallest interval containing the support of μ^* equals \mathbb{R}_{++} . Then one has*

$$(6) \quad \begin{aligned} & \{f \in \mathcal{L}_{b-}^0 : f \leq (\vartheta \cdot S)_T + g(S_T) \text{ for some } (\vartheta, g) \in \Theta\} \\ &= \{f \in \mathcal{L}_{b-}^0 : \langle f, \mu \rangle \leq 0 \text{ for all } \mu \in \mathcal{M}(\mathcal{Q})\} \end{aligned}$$

where $\mathcal{M}(\mathcal{Q})$ denotes the set of all martingale measures μ for S which satisfy $\mu_T := \mu \circ S_T^{-1} \in \mathcal{Q}$.

Remarks 11.

1. It follows from equation (6) that the set of all bounded attainable outcomes

$$\{f \in \mathcal{L}^\infty : f \leq (\vartheta \cdot S)_T + g(S_T) \text{ for some } (\vartheta, g) \in \Theta\}$$

is $\sigma(\mathcal{L}^\infty, ca)$ -closed. In general, the set of attainable outcomes under semistatic hedging is not closed, see [1]. Moreover, equation (6) implies that for each $f \in \mathcal{L}_{b-}^0$ and $m \in \mathbb{R}$ one has $f \leq m + (\vartheta \cdot S)_T + g(S_T)$ for some $(\vartheta, g) \in \Theta$ if and only if $\langle f, \mu \rangle \leq m$ for all $\mu \in \mathcal{M}(\mathcal{Q})$, which yields the superhedging duality

$$\inf \{m \in \mathbb{R} : m + (\vartheta \cdot S)_T + g(S_T) \geq f \text{ for some } (\vartheta, g) \in \Theta\} = \sup_{\mu \in \mathcal{M}(\mathcal{Q})} \langle f, \mu \rangle.$$

2. Even though \mathcal{Q} is dominated by the probability measure μ^* , one can check that the set of pricing measures $\mathcal{M}(\mathcal{Q})$ is not dominated in general.

3. If every $\mu \in \mathcal{Q}$ has the same barycenter $S_0 = \langle id, \mu \rangle \in \mathbb{R}_{++}$, then Proposition 10 holds for extended trading strategies $(\vartheta_1, \vartheta, g)$ with $\vartheta_1 \in \mathbb{R}$ and $(\vartheta, g) \in \Theta$.

4. If instead of the state space $\Omega = \mathbb{R}_{++}^T$ one considers $\Omega = [0, +\infty)^T$, Proposition 10 does not hold unless one allows ϑ_t to assume the value $+\infty$. To see this, let $T = 2$ and \mathcal{Q} be the convex hull of μ^* and $\{\xi_n d\lambda : n \in \mathbb{N}\}$, where $\mu^* := (\delta_0 + \xi d\lambda)/2$ for a strictly positive density ξ (w.r.t. the Lebesgue measure λ) with finite first moment and $\xi_n d\lambda \rightarrow d\mu^*$. It is possible to choose (ξ_n) such that \mathcal{Q} fulfills the assumptions of Proposition 10. Define $f := 1_{\{0\} \times (0,1)}$ so that $\langle f, \mu \rangle = 0$ for every $\mu \in \mathcal{M}(\mathcal{Q})$, and let (ϑ, g) such that $(\vartheta \cdot S)_T + g(S_T) \geq f$. We will see in the proof of the proposition that g has to be positive. Hence, whenever $\vartheta_2(0) \neq +\infty$ it follows that $g(x) \geq (1 - \vartheta_2(0)x) \vee 0$ for $x \in (0, 1)$ and therefore $\varphi(g) \geq \sup_n \int_0^1 ((1 - \vartheta_2(0)x) \vee 0) \xi_n(x) dx \geq 1/2$.

Proof of Proposition 10. The goal is to apply Proposition 6 to the set

$$H := \{h \in \mathcal{L}_{b-}^0 : h \leq (\vartheta \cdot S)_T + g(S_T) \text{ for some } (\vartheta, g) \in \Theta\}.$$

It is clear that H is monotone and contains 0, and we claim that H is normalized and tight. If $m \in H$ for some $m \geq 0$, then $m \leq g(x)$ for every $x \in \mathbb{R}_{++}$ since $(\vartheta \cdot S)_T = 0$ on the constant path $\omega = (x, \dots, x)$. Since $\varphi(g) \leq 0$ it follows that $m = 0$. To show that H is tight, fix $\varepsilon > 0$ and $n \in \mathbb{N}$. Due to compactness of the set \mathcal{Q} , one can show that there exist $\delta > 0$ such that $\mu((0, 2\delta]) \leq \varepsilon/(2n)$ for every $\mu \in \mathcal{Q}$. In combination with the assumption that $\lim_k \varphi((id - k)^+) = 0$, there thus exists $k \in \mathbb{N}$ such that $\varphi(g) \leq 0$ where $g(x) := nx1_{[k-1, \infty)}(x) + 2n1_{(0, 2\delta]}(x) - \varepsilon$. Define the stopping times $\tau := \inf\{t \geq 1 : S_t > k\}$ and $\sigma := \inf\{t \geq 1 : S_t < \delta\}$ as well as $\vartheta_t := -n1_{\{t \geq \tau+1\}} + n/\delta 1_{\{t \geq \sigma+1\}}$. Then $(\vartheta \cdot S)_T + g(S_T) \geq m1_{K^c} - \varepsilon$ for $K := [\delta, k]^T$ so that H is tight.

We next show that $\sup_n h_n \in H$ whenever (h_n) is an increasing sequence in H . Since $h_1 \in \mathcal{L}_{b-}^0$ there exists $c \in \mathbb{R}$ with $h_n \geq h_1 \geq c$. Let $(\vartheta^n, g^n) \in \Theta$ such that $(\vartheta^n \cdot S)_T + g^n(S_T) \geq h_n$. Considering the constant path $\omega = (x, \dots, x)$ it follows that $c \leq h_n(x, \dots, x) \leq g^n(x)$. Since $\langle g^n, \mu^* \rangle \leq \varphi(g^n) \leq 0$, we can apply the Komlós' theorem (see [9, Lemma A.1]) in order to obtain a sequence of forward convex combinations $\tilde{g}^n \in \text{conv}\{g^k : k \geq n\}$ which converge μ^* -almost surely to a Borel measurable function $g : \mathbb{R}_{++} \rightarrow [c, +\infty]$. By convexity of φ it holds $\varphi(\tilde{g}^n) \leq 0$ and by Fatou's lemma it follows that $\varphi(g) \leq 0$ so that the Borel set

$$C := \{x \in \mathbb{R}_{++} : \tilde{g}^n(x) \rightarrow g(x) \in \mathbb{R}\}$$

has μ^* -measure one. Redefine g to be $+\infty$ on the complement of this set. Passing to the same convex combinations used for \tilde{g}_n also for ϑ^n and h_n , it holds in obvious

notation that $\tilde{g}^n + (\tilde{\vartheta}^n \cdot S)_T \geq \tilde{h}_n$. For the purpose of readability we again write g^n , ϑ^n and h_n . Assume that there exists $x \in \mathbb{R}_{++}$ such that the sequence $(\vartheta_2^n(x))$ is not bounded. We focus on the case that $\limsup_n \vartheta_2^n(x) = +\infty$, the other case is treated analogously. Since $(\vartheta^n \cdot S)_T \geq c - g^n$, it follows for any path of the form $\omega = (x, y, \dots, y) \in \Omega$ with $y \in (0, x)$ that

$$-\infty = \liminf_n (\vartheta_2^n(x)(y - x)) = \liminf_n (\vartheta^n \cdot S)_T(\omega) \geq \liminf_n (c - g^n(y)).$$

However, since $(g^n(y))$ is bounded for $y \in C$ and $C \cap (0, x) \neq \emptyset$ by assumption, this already yields a contradiction. By induction it follows that (ϑ_t^n) is pointwise bounded for every t , so that $\vartheta_t := \limmed_n \vartheta_t^n$ is well-defined. By monotonicity of the medial limit it follows that

$$\sup_n h_n = \limmed_n h_n \leq \limmed_n ((\vartheta^n \cdot S)_T + g^n(S_T)) \leq (\vartheta \cdot S)_T + g(S_T)$$

which shows that $\sup_n h_n \in H$.

We finally show that H is regular, that is

$$a := \sup_{h \in H \cap U_b} \langle h, \mu \rangle = \sup_{h \in H \cap C_b} \langle h, \mu \rangle =: b$$

for every $\mu \in ca_+^1$. First notice that $a \geq b$ and since $\lambda h \in H$ for every $h \in H$ and $\lambda \in \mathbb{R}_+$, it follows that $a, b \in \{0, +\infty\}$. Let $\mu \in \mathcal{M}(\mathcal{Q})$, $h \in H$ (not necessarily upper semicontinuous) and $(\vartheta, g) \in \Theta$ such that $h \leq (\vartheta \cdot S)_T + g(S_T)$. It follows from a result on local martingales that $\langle (\vartheta \cdot S)_T, \mu \rangle = 0$ (see [13, Theorem 1 and Theorem 2]). Hence $\langle h, \mu \rangle \leq \varphi(g) \leq 0$, so that $a = b = 0$, and in particular $\mu \in H^\circ$. Conversely, let $\mu \notin \mathcal{M}(\mathcal{Q})$. First, if $\mu_T \notin \mathcal{Q}$, the hyperplane separation theorem yields the existence of a continuous bounded function $g: \mathbb{R}_{++} \rightarrow \mathbb{R}$ such that $\langle g, \mu_T \rangle > 0$ and $\varphi(g) \leq 0$. For $h := g \circ S_T \in H \cap C_b$ it follows that $b \geq \langle h, \mu \rangle > 0$, and therefore $b = +\infty$. Second, if $S_t \notin L^1(\mu)$ for some $t \in \{1, \dots, T\}$, then $b = +\infty$. Indeed,

$$(S_t - \varphi(id)) \wedge k \leq (\vartheta \cdot S)_T + g(S_T) \quad \text{for } g(x) := x - \varphi(id) \text{ and } \vartheta_s := -1_{\{s \geq t+1\}}$$

implies that $(S_t - \varphi(id)) \wedge k \in H \cap C_b$ and therefore $b \geq \langle (S_t - \varphi(id)) \wedge k, \mu \rangle \rightarrow +\infty$. Third, if S is not a martingale under μ , then there exists $t \in \{1, \dots, T\}$ and a continuous function ξ of the first $t-1$ components with values in $[-1, 1]$ such that $\varepsilon := \langle \xi \cdot (S_t - S_{t-1}), \mu \rangle > 0$, where, by integrability of $S_t - S_{t-1}$, we may assume that ξ has support $(0, k]^{t-1}$. Define

$$f := (-n) \vee (\xi \cdot (S_t - S_{t-1})) \wedge n \in C_b,$$

and the strategy (ϑ_s) as 0 if $s < t$, ξ if $s = t$, and $-1_{\{S_t \geq n-k\}}$ if $s \geq t+1$, as well as $g(x) := x1_{[n-k, \infty)}(x) - \varepsilon/2$. For $n \in \mathbb{N}$ large enough one has $\varphi(g) \leq 0$, and since

$$(\vartheta \cdot S)_T + g(S_T) = S_T 1_{\{S_T \geq n-k\}} + \xi \cdot (S_t - S_{t-1}) + (S_t - S_T) 1_{\{S_t \geq n-k\}} - \varepsilon/2$$

and the fact that $S_t \geq n-k$ whenever $\xi \cdot (S_t - S_{t-1}) \leq -n$, it follows that $f - \varepsilon/2 \leq (\vartheta \cdot S)_T + g(S_T)$ and therefore $\langle f - \varepsilon/2, \mu \rangle > 0$, which shows that $b > 0$ and consequently $b = +\infty$. Hence, for $\mu \notin \mathcal{M}(\mathcal{Q})$ it holds $\sup_{h \in H} \langle h, \mu \rangle = +\infty$, so that $\mu \notin H^\circ$. In summary, one has $H^\circ = \{\lambda \mu : \lambda \in \mathbb{R}_+, \mu \in \mathcal{M}(\mathcal{Q})\}$.

In view of Proposition 6 we conclude that $f \in H$ if and only if $\langle f, \mu \rangle \leq 1$ for all $\mu \in H^\circ$ if and only if $\langle f, \mu \rangle \leq 0$ for all $\mu \in \mathcal{M}(\mathcal{Q})$. \square

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